LESSONS LEARNED FROM LESSONS TAUGHT

MARK TOMFORDE

ABSTRACT. This is a transcript of the speech I gave on January 17, 2020 after receiving the Haimo Award. In it I discuss how my personal views on the teaching and learning of mathematics have evolved over time. In particular, I share how certain beliefs I once held as a beginning instructor have changed and been replaced by new viewpoints or more nuanced ways of thinking about students.

In my twenty years of teaching I have spent a great deal of time thinking about my personal teaching philosophy and my goals in the classroom. As with many instructors, my ideas and attitudes towards teaching have changed over time. In this brief talk today I want to share with you the ways that those beliefs have evolved. In particular, I want to tell you about four beliefs I had when I started teaching that have changed significantly.

In some cases, I've concluded my original beliefs were wrong and replaced them with completely different notions. Other times, I've found the beliefs not so much wrong as incomplete, and I've adjusted them to develop more inclusive and nuanced viewpoints that better serve my students. Here are the four beliefs:

Old Belief #1: My main objective is to teach mathematical content. And, as a corollary, the more content I cover — or the further I can get through the textbook — the better.

New Belief #1: My main objective is to teach my students to think.

The more I taught, the more I found myself asking "In 10 years, what is it that students will remember from this course?" Related to this question, "What is it that I *want* them to remember in 10 years time?" After all, when I think of my goals for my classes, it is not merely to prepare students for careers, but to prepare them for life.

Most likely, many of my students will not remember the chain rule, or that uniform convergence implies pointwise convergence. Indeed, many of them may not even use the theorems from class in their careers or lives. Instead, what they will remember are the parts of the class that they use regularly. So what are these parts? More important than the facts themselves are the ways of thinking developed when students engage those facts. This includes logic, problem solving, attention to detail, skills for thinking abstractly, and the ability to effectively communicate technical ideas. These skills are highly transferrable to a variety of situations, both in mathematics and in life. Moreover, they all contribute to habits of mind that students will use daily, and they shape the way that students view the world. In short, it is these critical thinking skills that are the heart of what I wish to teach.

Yes, the mathematical content is still important, but I now make a great deal of effort to choose and shape the mathematical content to convey the more important ways of thinking that underly the mathematics. As a result, I worry less about exactly which facts I cover in class or about finishing the syllabus at the expense of students not keeping up. Rather, I focus more on teaching students to think, and remind myself that if they know how to think they know how to learn. And if they know how to learn, they can later teach themselves any particular facts I may not cover in class.

Therefore, I design my courses to teach students to think critically, to communicate effectively, and to solve problems. Indeed, I would rather that students think carefully and deeply about a few (ostensibly) simple topics, than be exposed to an abundance of results providing familiarity at the expense of understanding.

Old Belief #2: The more students I can reach, the better.

New Belief #2: Quality is more important than quantity.

When we as human beings evaluate effectiveness, we like to reduce performance to a single metric, ideally something we can count and compare by simply asking "Which is more?". As much as we'd like to think otherwise, Mathematicians are no different. Want to determine the better researcher? Count number of publications on CVs or number of students supervised. Want to determine which Math Circle is having a larger impact? Count the number of students attending. When we think about this for a moment, we realize it is flawed reasoning. Yet it is all too easy for us to fall into this trap. When I supervise an undergraduate research project, I am tempted to think that if I supervise more students, then I am doing a better job. When I run a high school outreach program, I am tempted to think the more high schoolers that attend, the better. And even if I don't believe it myself, I know that most likely that is how my impact will be evaluated by others.

But, when I pause to consider the situation more carefully, I find it more accurate to think of myself and what I have to offer as a fixed quantity, and every additional student I add divides my time, attention, and effort so that each gets less. Likewise, when I think of the number of projects I take on, I acknowledge that with every additional commitment I have less of myself that I can devote to the others.

As a result I've decided it is better to adopt the approach of "fewer but better". I now make a conscious effort to only commit to a handful of projects, but give each a great deal of attention. Likewise, I focus on meaningful, authentic interactions with students. When I've run a math circle, there are situations where I choose to have six middle students involved rather than twenty. And when I've supervised research projects, I've turned down requests from multiple students so that I can focus on only one or two.

Initial Belief #3: The best students are our future; teach to them.

New Belief #3: I should direct my teaching to all students, paying particular attention to those that are struggling.

We sometimes view ourselves as teaching future mathematicians and training the next generation of mathematics professors. While there is some truth in this, it is an incomplete view of what we should accomplish in our classes, and it can lead to dangerous norms. For example, we may grow to believe our best students are more important and take precedence over the others. We may excuse problem behavior from students if we consider them to "have talent". And worst of all, we may overlook students with unnoticed potential and lose people of value.

We like to think that we as faculty are good at identifying potential, but the reality is we are all horrible at it. I have seen countless instances of students that have been overlooked by faculty, yet have gone on to make valuable contributions. I only wonder how many students like them were unable to persist.

Therefore, we need to change our mindset. Rather than viewing our mission as "Finding Talent", we should think of it as "Developing Talent". It is easy to get caught up interacting with the best and brightest students, and it is exciting to challenge them and provide enrichment opportunities. But, it is equally as important — and I would argue even more rewarding — to connect with the entire spectrum of students. By directing effort towards all and nurturing everyone in the classroom, a teacher can cultivate an atmosphere in which average students will blossom into great students, and struggling students will persist rather than depart.

These seemingly minor victories are often of critical importance and great benefit to mathematics and society as a whole. The good students will find a way to succeed on their own, but the average or struggling students are the ones that most need the guidance, encouragement, and support that can be provided by teachers and mentors.

Initial Belief #4: Good students are the ones who can prove theorems, solve problems, and do well in the classroom.

New Belief #4: There are many ways for students to be successful and contribute to mathematics.

In mathematics we often promote a narrow definition of career success; typically that of a tenure-track job at a research institution with a light teaching load. We often convey this to our students — explicitly and implicitly — at both the undergraduate and graduate levels. However, mathematics requires a much richer ecosystem than simply a large number of researchers proving theorems while doing their best to minimize teaching responsibilities. Consequently, there is danger in the narrow definition of "success" that is often communicated in mathematics departments.

When I teach my students, particularly the math majors, I regard them as the future of mathematics — in all its forms — and I let them know that they will be responsible for the manner in which the subject will be viewed, used, and supported by society. I am fond of telling my students that I want them to become "stewards of the discipline". This is an idea I first learned about from the following quotation:

> "The purpose of doctoral education, taken broadly, is to educate and prepare those to whom we can entrust the vigor, quality, and integrity of the field . . . Someone who will creatively generate new knowledge, critically conserve valuable and useful ideas, and responsibly transform those understandings through writing, teaching, and application. We call such a person a 'steward of the discipline'."
>
> — Chris M. Golde, Carnegie Foundation for the Advancement of Teaching, published in the February 2012 issue of the *AMS Notices*

This idea of stewardship is predicated on the notion that there are many ways for individuals to contribute to mathematics, and moreover that mathematics needs a variety of participants with different talents and skills in order to flourish and remain viable as a discipline. In addition to researchers, mathematics requires teachers, advocates, public relations officers, writers, and various others to lead, serve, maintain, apply, and expand the subject.

In conclusion, I want to tell you about a specific technique I use to help students to become "stewards of the discipline". Specifically, I encourage them to develop what I sometimes refer to as a "Grand Project". I tell my students that stewardship is an ethic that embodies responsibility and care. Stewardship involves not only maintaining the community of mathematics, but contributing to it — and there are countless ways to contribute. The best way is to find a need that you have the ability to address, and then contribute to filling that need. Some ideas for Grand Projects: start a Math Circle or tutoring program for middle schoolers, write a guide for incoming students in your department, start a department newsletter, organize a regular department picnic . . . or better yet, come up with your own ideas.

I tell the students that a Grand Project is something special that you choose to work on to make a personal connection with the mathematics community. It can also be something you work on to revitalize yourself when you are tired from classes or studying. I've found that these projects provide students with a sense of connection and purpose. In my experience the greatest effects are often seen in students from underrepresented groups. A Grand Project can help them to remember why they are working so hard to study math, and it can remind them that what they are learning can be used to help other people. In doing so, it deepens their commitment to mathematics.

These are just a few of the ways my views on teaching have evolved over my career, and I thank you for taking the time to listen to me today. I wish to leave you with one final quotation.

> "Not everyone can be famous but everyone can be great, because greatness is determined by service . . . You only need a heart full of grace and a soul generated by love."
> — Martin Luther King, Jr.