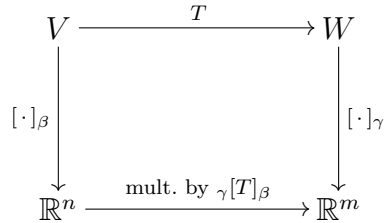


Please do the following four problems and hand them in on April 13.



Problem 1: Define $T : \mathbb{P}_2(t) \rightarrow M_{2 \times 2}(\mathbb{R})$ by

$$T(p(t)) = \begin{pmatrix} p(0) & p(1) \\ 0 & p'(0) \end{pmatrix}$$

(where p' denotes the derivative of p). Let $\beta = \{t^2, t, 1\}$ be a basis for $\mathbb{P}_2(t)$, and let $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis for $M_{2 \times 2}(\mathbb{R})$.

- (1 point) Calculate ${}_{\gamma}[T]_{\beta}$. What size is this matrix?
- (0.5 points) Let $\vec{x} = 3t^2 - 2t + 5$. Calculate $[\vec{x}]_{\beta}$ and $[T(\vec{x})]_{\gamma}$.
- (0.5 points) Show that ${}_{\gamma}[T]_{\beta} [\vec{x}]_{\beta} = [T(\vec{x})]_{\gamma}$.

Problem 2: Define $T : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ by

$$T(A) = A^T.$$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis for $M_{2 \times 2}(\mathbb{R})$.

- (1 point) Calculate $[T]_{\beta}$. What size is this matrix?
- (0.5 points) For an arbitrary $\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, calculate $[\vec{x}]_{\beta}$ and $[T(\vec{x})]_{\beta}$.
- (0.5 points) Show that $[T]_{\beta} [\vec{x}]_{\beta} = [T(\vec{x})]_{\beta}$.

Problem 3: Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$T(\vec{x}) = A\vec{x}$$

where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$ be a basis for \mathbb{R}^2 .

- (a) (1 point) Calculate $[T]_\beta$. What size is this matrix?
- (b) (0.5 points) For $\vec{x} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, calculate $[\vec{x}]_\beta$ and $[T(\vec{x})]_\beta$.
- (c) (0.5 points) Show that $[T]_\beta [\vec{x}]_\beta = [T(\vec{x})]_\beta$.
- (d) (1 point) Find an invertible matrix P such that $A = P [T]_\beta P^{-1}$. Verify that, in fact, $P [T]_\beta P^{-1}$ is equal to A . What are the columns of P ?

Problem 4: Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$T(\vec{x}) = A\vec{x}$$

where $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$. Let $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ be a basis for \mathbb{R}^3 .

- (a) (1 point) Calculate $[T]_\beta$. What size is this matrix?
- (b) (0.5 points) For $\vec{x} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$, calculate $[\vec{x}]_\beta$ and $[T(\vec{x})]_\beta$.
- (c) (0.5 points) Show that $[T]_\beta [\vec{x}]_\beta = [T(\vec{x})]_\beta$.
- (d) (1 point) Find an invertible matrix P such that $A = P [T]_\beta P^{-1}$. Verify that, in fact, $P [T]_\beta P^{-1}$ is equal to A . What are the columns of P ?

Notice that in Problem 4 the basis β is chosen so that $[T]_\beta$ is a diagonal matrix. In this case, β gives a coordinate system in which T has a matrix representation $[T]_\beta$ that is very easy to work with.