Please do the following four problems and hand them in on April 13.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\,\cdot\,]_\beta}"'] & W \arrow[d, "{[\,\cdot\,]_\gamma}"] \\ \mathbb{R}^n \arrow[r, "\text{mult. by } {}_\gamma[T]_\beta"] & \mathbb{R}^m \end{tikzcd}$$

Problem 1: Define $T : \mathbb{P}_2(t) \to M_{2\times 2}(\mathbb{R})$ by

$$T(p(t)) = \begin{pmatrix} p(0) & p(1) \\ 0 & p'(0) \end{pmatrix}$$

(where p' denotes the derivative of p). Let $\beta = \{t^2, t, 1\}$ be a basis for $\mathbb{P}_2(t)$, and let $\gamma = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ be a basis for $M_{2\times 2}(\mathbb{R})$.

(a) (1 point) Calculate ${}_\gamma[T]_\beta$. What size is this matrix?

(b) (0.5 points) Let $\vec{x} = 3t^2 - 2t + 5$. Calculate $[\vec{x}]_\beta$ and $[T(\vec{x})]_\gamma$.

(c) (0.5 points) Show that ${}_\gamma[T]_\beta\ [\vec{x}]_\beta = [T(\vec{x})]_\gamma$.

Problem 2: Define $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by

$$T(A) = A^T.$$

Let $\beta = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ be a basis for $M_{2\times 2}(\mathbb{R})$.

(a) (1 point) Calculate $[T]_\beta$. What size is this matrix?

(b) (0.5 points) For an arbitrary $\vec{x} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, calculate $[\vec{x}]_\beta$ and $[T(\vec{x})]_\beta$.

(c) (0.5 points) Show that $[T]_\beta\ [\vec{x}]_\beta = [T(\vec{x})]_\beta$.

Problem 3: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\vec{x}) = A\vec{x}$$

where $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$. Let $\beta = \{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)\}$ be a basis for $\mathbb{R}^2$.

(a) (1 point) Calculate $[T]_\beta$. What size is this matrix?

(b) (0.5 points) For $\vec{x} = \left(\begin{smallmatrix} 6 \\ -2 \end{smallmatrix}\right)$, calculate $[\vec{x}]_\beta$ and $[T(\vec{x})]_\beta$.

(c) (0.5 points) Show that $[T]_\beta \, [\vec{x}]_\beta = [T(\vec{x})]_\beta$.

(d) (1 point) Find an invertible matrix P such that $A = P\,[T]_\beta\,P^{-1}$. Verify that, in fact, $P\,[T]_\beta\,P^{-1}$ is equal to A. What are the columns of P?

Problem 4: Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T(\vec{x}) = A\vec{x}$$

where $A = \left(\begin{smallmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{smallmatrix}\right)$. Let $\beta = \left\{\left(\begin{smallmatrix} 1 \\ 2 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right)\right\}$ be a basis for $\mathbb{R}^3$.

(a) (1 point) Calculate $[T]_\beta$. What size is this matrix?

(b) (0.5 points) For $\vec{x} = \left(\begin{smallmatrix} -1 \\ -1 \\ 2 \end{smallmatrix}\right)$, calculate $[\vec{x}]_\beta$ and $[T(\vec{x})]_\beta$.

(c) (0.5 points) Show that $[T]_\beta \, [\vec{x}]_\beta = [T(\vec{x})]_\beta$.

(d) (1 point) Find an invertible matrix P such that $A = P\,[T]_\beta\,P^{-1}$. Verify that, in fact, $P\,[T]_\beta\,P^{-1}$ is equal to A. What are the columns of P?

Notice that in Problem 4 the basis β is chosen so that $[T]_\beta$ is a diagonal matrix. In this case, β gives a coordinate system in which T has a matrix representation $[T]_\beta$ that is very easy to work with.