An Extra Worksheet involving the Well-Ordering Principle

Some students asked about how the well-ordering principle can be used in place of mathematical induction to prove statements. I put this worksheet together to explain the process in more detail. This is extra information for your own edification. Nothing from it needs to be turned in, and you're not responsible for this material in future homework or exams.

The Well-Ordering Principle: "Every nonempty subset of the natural numbers contains a smallest element."

Note that the well-ordering principle is equivalent to the following: "Any subset of the natural numbers without a smallest element is empty."

As we saw in class, the well-ordering principle is equivalent to the Principle of Mathematical Induction. Like induction, the well-ordering principle can be used to prove that a collection of statements indexed by the natural numbers is true.

USING THE WELL-ORDERING PRINCIPLE IN PROOFS

Let $P(n)$ be a statement involving a natural number n. To prove $(\forall n \in \mathbb{N})P(n)$, begin by letting $S := \{n \in \mathbb{N} : P(n) \text{ is false}\}$.

Next, argue that there is no smallest element of S by doing a proof by contradiction: Suppose that S has a smallest element n. (Note that n is the smallest natural number for which $P(n)$ is false.) We obtain a contradiction by following the following two steps.

Step 1: Show that $n \neq 1$ by showing $P(1)$ is true.

Step 2: Show that n cannot be a natural number greater than 1 by proving

$$\sim P(n) \implies \sim P(n-1)$$

which contradicts that n is the smallest natural number for which $P(n)$ is false.

Finally, conclude that since S is a subset of the natural numbers with no smallest element, the well-ordering principle implies that $S = \emptyset$, and hence $P(n)$ is true for all $n \in \mathbb{N}$.

Note that the similarity of the two steps above to the method of proof using the Principle of Mathematical Induction. In a proof involving mathematical induction we verify the following:

Base Case: Show that $P(1)$ is true.

Inductive Step: Show that if $n \geq 2$, then $P(n-1) \implies P(n)$.

(In the inductive step we more typically show that if $n \in \mathbb{N}$, then $P(n) \implies P(n+1)$. However, we see that this is the same as above, simply by using n in place of $n-1$ for the variable.)

We see that Step (1) in the box above is exactly the same as the Base Case, and Step (2) is the contrapositive of (and hence logically equivalent to) the Inductive Step. This further illustrates that the well-ordering principle and the principle of mathematical induction are equivalent.

Since the steps involved in using the well-ordering principle involve the contrapositive, it is often best to use this method when establishing statements involving a negation or that assert something "does not occur". Here is such a problem in which you can practice the method above.

Problem: Using the well-ordering principle, prove that if $n \in \mathbb{N}$, then 2 does not divide $3n^2+5n-1$.