1. Notes on Relations

1.1 Cartesian Products

Definition 1.1 If A and B are sets, the *cartesian product* (or simply *product*) of A and B is the set

$$A \times B := \{(x,y) : x \in A \text{ and } y \in B\}.$$

Thus $A \times B$ is the set of all ordered pair having an element of A in the first coordinate of the pair and an element of B is the second element of the pair. Note that the ordering matters, so that (x,y) is considered distinct from (y,x). In fact, if $(x,y),(z,w) \in A \times B$, then $(x,y) = (z,w)$ is and only if $x = z$ and $y = w$.

■ **Example 1.2** Let $A := \{a,b\}$ and $B := \{1,2,3\}$. Then

$$A \times B = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}.$$

■

Note that if A is a finite set containing m elements and B is a finite set containing n elements, then $A \times B$ is a finite set containing mn elements.

■ **Example 1.3** Let $A = B = \mathbb{R}$. Then

$$\mathbb{R} \times \mathbb{R} = \{(x,y : x,y \in \mathbb{R}\}$$

is the set of all ordered pairs of real numbers. The set $\mathbb{R} \times \mathbb{R}$ is often called the *cartesian plane*, and it is common to denote it by $\mathbb{R}^2$. ■

★ We can extend the definition of the cartesian product to three or more sets in the obvious way. For example, if A, B, and C are sets, we define

$$A \times B \times C := \{(x,y,z) : x \in A, y \in B, \text{ and } z \in C\}.$$

Thus $A \times B \times C$ consists of ordered 3-tuples. Technically speaking $(A \times B) \times C$ is not exactly the same as $(A \times (B \times C)$, because elements of the first have the form $((x,y),z)$ while elements of the second have the form $(x,(y,z))$. However, we can easily identify either form with (x,y,z) by ignoring parentheses in the obvious way. Thus we often identify $(A \times B) \times C$ and $(A \times (B \times C)$ with $A \times B \times C$, thinking of all three as being "the same".

Given two sets A and B, there are situations in which we wish to say that certain elements of A are "related" to certain elements of B. In other words, we wish to identify certain pairs (x,y) with $x \in A$ and $y \in B$ and say that x is related to y. In other words, we are simply choosing a subset of $A \times B$ to specify the related pairs. The following definition makes this precise and establishes some notation

Definition 1.4 Let A and B be sets. A *relation* from A to B is a subset $R \subseteq A \times B$. A relation from A to A is often called a *relation on* A.

If $(x,y) \in R$, we write xRy and say x *is related to* y. Note that the symbol R is doing double duty" R is both the subset of $A \times B$ and R is used in the notation xRy to denote "related to".

Example 1.5 Let $A = \{1,2,3\}$. We may define a relation on A by $R := \{(1,2),(1,3),(2,3)\}$. Note that x is related to y (equivalently $(x,y) \in R$) precisely when x is strictly less than y. Thus the relation R is the familiar relation $<$, and xRy if and only if $x < y$. ■

Example 1.6 Suppose there are five math classes offered this semester: Real Analysis, Complex Analysis, Abstract Algebra, Topology, and Geometry. Suppose also there are four students enrolled in these courses as follows:

- Alex is enrolled in Real Analysis and Complex Analysis
- Beatrice is enrolled in Complex Analysis, Abstract Algebra, and Topology
- Cynthia is enrolled in Topology.
- David is not enrolled in any courses this semester.

Let

$$A := \{\text{Alex}, \text{Beatrice}, \text{Cynthia}, \text{David}\}$$

be the set of students, and let

$$B := \{\text{Real Analysis}, \text{Complex Analysis}, \text{Abstract Algebra}, \text{Topology}, \text{Geometry}\}$$

be the set of math classes offered this semester. We may define a relation

$$R := \{(x,y) \in A \times B : \text{student } x \text{ is enrolled in class } y\}.$$

If we wish to write out the elements of R, they are

$$R = \{(\text{Alex, Real Analysis}), (\text{Alex, Complex Analysis}), (\text{Beatrice, Complex Analysis}), (\text{Beatrice, Abstract Algebra}), (\text{Beatrice, Topology}), (\text{Cynthia, Topology})\}.$$

Moreover, we can make statements such as "Cynthia is related to Topology" or using our notation: Cynthia R Topology. Note that for any student x there may no, one, or multiple classes y such that xRy, and for any class y there may be no, one, or multiple students x such that xRy. ■

Example 1.7 Let A be a set. We may define a relation on A by $R := \{(x,x) : x \in A\}$. Then xRy if and only if $x = y$. Thus the relation R is the familiar relation $=$. We call R the *identity* (or *equality*) relation. ■

★ To define a relation from A to B, we must specify a subset $R \subseteq A \times B$ and we then write aRb when $(a,b) \in R$. However, one can also specify the subset by describing when an element $a \in A$ is related to an element $b \in B$, and then defining the subset to be the collection of all ordered pairs $(a,b) \in A \times B$ such that a is related to b.

For example, suppose we define a relation R from $\mathbb{N}$ to $\mathbb{Z}$ by "xRy if and only if $x = y^3$". This specifies a subset of $\mathbb{N} \times \mathbb{Z}$ by

$$\{(x,y) \in \mathbb{N} \times \mathbb{Z} : x = y^3\} = \{(1,1),(8,2),(27,3),(64,4),\ldots\}.$$

Note that in many situations it is easier to define a relation by describing when x is related to y rather than writing out the subset explicitly.

Relations are incredibly general, and as such there is not a great deal we can say about *all* relations. Instead, we shall study special types of relations, such as equivalence relations and order relations, both of which shall introduce in the coming sections. The general notion of a "relation" is useful for providing a unifying framework in which we can view specific kinds of relations as special cases of one general object. In practice, however, different kinds of relations behave in fairly different ways, so for the most part we have to study them individually.

1.2 Equivalence Relations

In order to study specific kinds of relations, we will shall consider various properties that a relation can satisfy.

Definition 1.8 Let A be a set, and let R be a relation on A. We define the following properties for the relation R:

Reflexive: For all $x \in A$, we have xRx.
Symmetric: For all $x,y \in A$, if xRy, then yRx.
Transitive: For all $x,y,z \in A$, if xRy and yRz, then xRz.

Definition 1.9 Let A be a set, and let R be a relation on A. We call R an *equivalence relation* if R is reflexive, symmetric, and transitive. When R is an equivalence relations, two elements of A that are related by R are called *equivalent*. When R is an equivalence relation it is common to use symbols such as $\equiv$, $\sim$, $\cong$, or $\approx$ in place of R.

■ **Example 1.10** Let A be a nonempty set of people. We shall assume that all people in A have one biological father and one biological mother. Define the following relations.

- xR_1y if and only if $x = y$ or x is the parent of y.
- xR_2y if and only if x is the parent of y or y is the parent of x
- xR_3y if and only if x is an ancestor of y.
- xR_4y if and only if x and y have a common ancestor.
- xR_5y if and only if x is an ancestor or a descendant of y.
- xR_6y if and only if $x = y$ or x is an ancestor of y.
- xR_7y if and only if x and y have the same father and the same mother.

Let us consider which of the properties of Definition 1.8 hold for each relation: Relation R_1 is reflexive, but not symmetric and not transitive. Relation R_2 is symmetric, but not reflexive and not transitive. Relation R_3 is transitive, but not reflexive (i.e., a person is not their own ancestor) and not symmetric. Relation R_4 is reflexive and symmetric, but not transitive. Relation R_5 is symmetric and transitive, but not reflexive. Relation R_6 is reflexive and transitive, but not symmetric. Relation R_7 is reflexive, symmetric, and transitive. Thus R_7 is the only relation that is an equivalence relation. ■

■ **Example 1.11** For any set A, the equality relation $=$ is reflexive, symmetric, and transitive. Hence $=$ is an equivalence relation. ■

★ Equality may be considered the prototypical example of an equivalence relations. For any equivalence relation, we consider elements that are related via this relation to be "the same" even if they are not equal.

■ **Example 1.12** Define a relation $\equiv_{\text{ab}}$ on $\mathbb{R}$ by

$$x \equiv_{\text{ab}} y \qquad \text{if and only if} \qquad |x| = |y|.$$

Let us show that $\equiv_{\text{ab}}$ is an equivalence relation.
Reflexive: Let $x \in \mathbb{R}$. Then $|x| = |x|$, so $x \equiv_{\text{ab}} x$.
Symmetric: Let $x, y \in \mathbb{R}$ and suppose that $x \equiv_{\text{ab}} y$. Then $|x| = |y|$, and hence $|y| = |x|$, and $y \equiv_{\text{ab}} x$.
Transitive: Let $x, y, z \in \mathbb{R}$ and suppose that $x \equiv_{\text{ab}} y$ and $y \equiv_{\text{ab}} z$. Then $|x| = |y|$ and $|y| = |z|$, from which it follows that $|x| = |z|$, and $x \equiv_{\text{ab}} z$.

■

■ **Example 1.13** Fix $n \in \mathbb{N}$, and consider the relation "equivalence modulo n" on $\mathbb{Z}$, given by

$$x \equiv y \pmod{n} \qquad \text{if and only if} \qquad n \,|\, y - x.$$

Let us show that equivalence modulo n is an equivalence relation.
Reflexive: Let $x \in \mathbb{R}$. Then $x - x = 0$, and hence $n \,|\, x - x$, so $x \equiv x \pmod{n}$.
Symmetric: Let $x, y \in \mathbb{R}$ and suppose that $x \equiv y \pmod{n}$. Then $n \,|\, y - x$, so $n \,|\, -(y - x)$, and $n \,|\, x - y$. Hence $y \equiv x \pmod{n}$.
Transitive: Let $x, y, z \in \mathbb{R}$ and suppose that $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$. Then $n \,|\, y - x$ and $n \,|\, z - y$. Since $z - x = (z - y) + (y - x)$, we see that $n \,|\, z - x$, and hence $x \equiv z \pmod{n}$. ■

■ **Example 1.14** Consider the set $F := \{(a,b) : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$ and define a relation $\sim$ on F by

$$(a,b) \sim (c,d) \qquad \text{if and only if} \qquad ad = cb.$$

Let us show that the relation $\sim$ is an equivalence relation.
Reflexive: Let $(a,b) \in F$. Then $ab = ab$, so $(a,b) \sim (a,b)$.
Symmetric: Let $(a,b), (c,d) \in F$, and suppose that $(a,b) \sim (c,d)$. Then $ad = cb$. Hence $cb = ad$, so that $(c,d) \sim (a,b)$.
Transitive: Let $(a,b), (c,d), (e,f) \in F$ and suppose that $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $ad = cb$ and $cf = ed$. Thus

$$(af)(bd) = (ad)(fb) = (cb)(fb) = (cf)b^2 = (ed)b^2 = (eb)(bd)$$

and since $b \neq 0$ and $d \neq 0$, we may cancel to obtain $af = eb$. Thus $(a,b) \sim (e,f)$.

Notice that motivation for this example comes from fractions. A fraction, such as $\frac{1}{2}$ may be written as a fraction in many different ways: $\frac{1}{2} = \frac{2}{4} = \frac{-3}{-6} = \frac{500}{1000} = \ldots$. Given a fraction $\frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$, we may identify this fraction with the pair (a,b) in our set above. If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, we see that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = cb$. Thus the relation $\sim$ may viewed as describing when two representations of a fraction are the same. ■

1.2.1 Equivalence Classes

Equivalence relations generalize the notion of equality. When two elements are related via an equivalence relation, we may think of them as being "the same" even if they are not equal. For example, if we consider the equivalence relation $\bmod 3$ on $\mathbb{Z}$, then we see that 1 is equivalent to 4, and thus we may think of 1 and 4 as being "the same". Moreover, we can consider

$$\{\ldots, -5, -2, 1, 4, 7, \ldots\}$$

the set of all elements equivalent to 2. Each element in this set is considered to be "the same" as 2 (at least as far as the relation $\bmod 3$ is concerned), and we call this set the "equivalence class" of 2 with respect to the equivalence relation $\bmod 3$. This motivates the following definition.

Definition 1.15 Let R be an equivalence relation on a set A. For any $a \in A$, the *equivalence class* of a is the set

$$[a] := \{b \in A : aRb\}.$$

Other common notations for $[a]$ include $\overline{a}$ and a/R. Note that for $x, y \in A$, we have $[x] = [y]$ if and only if xRy.

The set

$$A/R := \{[a] : a \in A\}$$

of all equivalence classes is called A *modulo* R (or A *mod* R for short). Note that if $X \in A/R$ is an equivalence class, then $X = [a]$ for some $a \in A$. The element a is called a *representative* for the equivalence class X. Representatives are not unique, however any two representatives for a given equivalence class are equivalent; i.e., if $X = [a] = [b]$, then aRb.

■ **Example 1.16** If A is a set, equality $=$ is an equivalence relation. For any $a \in A$, the equivalence class of a is simply the singleton set containing a; that is, $[a] = \{a\}$. Thus $A/=$ is the collection $\{\{a\} : a \in A\}$ of singleton sets that contain each element of A. ■

■ **Example 1.17** Consider the set $\mathbb{Z}$ with the equivalence relation $\equiv_{\text{ab}}$ defined by $x \equiv_{\text{ab}} y$ if and only if $|x| = |y|$. We see that 0 is equivalent only to 0, so that $[0] = \{0\}$. However, for any integer $x \neq 0$, we see that both x and $-x$ are equivalent to x, so that $[x] = \{-x, x\}$. Thus $\mathbb{Z}/\equiv_{\text{ab}}$ is equal to

$$\{\{0\}, \{-1, 1\}, \{-2, 2\}, \{-3, 3\}, \ldots\}.$$

■

■ **Example 1.18** Fix $n \in \mathbb{N}$, and consider $\mathbb{Z}$ with the equivalence relation $\operatorname{mod} n$. We see that

$$\begin{aligned}
[0] &= \{\ldots -2n, -n, 0, n, 2n, \ldots\} \\
[1] &= \{\ldots -2n+1, -n+1, 1, n+1, 2n+1, \ldots\} \\
[2] &= \{\ldots -2n+2, -n+2, 2, n+2, 2n+2, \ldots\} \\
&\vdots \\
[n-1] &= \{\ldots -n-1, -1, n-1, 2n-1, 3n-1, \ldots\}.
\end{aligned}$$

Furthermore, every natural number is in one of the above equivalence classes: If $m \in \mathbb{N}$, then using the division algorithm, there exist $q \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, n-1\}$ such that $m = qn + r$. It follows that $m \equiv r \pmod{n}$, so that $m \in [r]$ and $[m] = [r]$. Since $[r]$ is one of the equivalence classes listed above, we see that m is in one of these equivalence classes, and that this equivalence class is equal to $[m]$. Therefore, we may conclude $\mathbb{Z}/\operatorname{mod} n$ is equal to $\{[0], [1], [2], \ldots, [n-1]\}$. ■

1.2.2 Partitions

If we look at the equivalence classes from the prior examples, we notice an interesting phenomenon: The equivalence classes subdivide the entire set into non-overlapping pieces each of which represents distinct (i.e., non-equivalent) elements. For example, with the relation $\operatorname{mod} n$ on $\mathbb{Z}$, we saw that the equivalence classes subdivided $\mathbb{Z}$ into n distinct classes given by $[0], [1], \ldots, [n-1]$. Elements in each class are equivalent (i.e., the same) and when we identify them (i.e., group them together without distinguishing between equivalent elements) we subdivide $\mathbb{Z}$ to form $\mathbb{Z}/\operatorname{mod} n = \{[0], [1], \ldots, [n-1]\}$, which has n elements. This subdividing is known as a *partition* of the set A, and the following definition makes the notion precise.

Definition 1.19 Let A be a nonempty set. A *partition of* A is a collection $\mathscr{P}$ of subsets of A satisfying the following three conditions:

(i) If $X \in \mathscr{P}$, then $X \neq \emptyset$.
(ii) if $X, Y \in \mathscr{P}$, then either $X = Y$ or $X \cap Y = \emptyset$.
(iii) $\bigcup_{X \in \mathscr{P}} X = A$.

A collection satisfying the second condition in Definition 1.19 is called *pairwise disjoint*. Thus a partition of A may be described as a collection of nonempty, pairwise disjoint subsets of A whose union is A.

It turns out that equivalence classes of equivalence relations form a partition. Conversely, any partition of a set is the collection of equivalence classes of some equivalence relation. Thus partitions of a set correspond to equivalence relations on that set, with each partition given by the equivalence classes of the equivalence relation. The following theorem makes this precise.

Theorem 1.20 Let A be a set. If R is an equivalence relation on A, then the equivalence classes of R form a partition of A. Conversely, if $\mathscr{P}$ is a partition of A, then there exists an equivalence relation R such that $A/R = \mathscr{P}$.

Proof. Let R be an equivalence relation on A. We shall shows that A/R, the collection of equivalence classes of R, forms a partition of A. To do so, we verify A/R satisfies the three conditions of Definition 1.19.

To begin, observe that if $[a] \in A/R$, then by the reflexivity of R we have aRa. Thus $a \in [a]$, and $[a] \neq \emptyset$. Hence A/R is a collection of nonempty sets.

Next, we shall show that the collection A/R is pairwise disjoint. It suffices to show that if $[a],[b] \in A/R$ and $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$. To this end, let $[a],[b] \in A/R$ with $[a] \cap [b] \neq \emptyset$. Then there exists $c \in [a] \cap [b]$. Since $c \in [a]$, we have cRa, and since $c \in [b]$, we have cRb. Because cRa and cRb, symmetry and transitivity of R imply that aRb. It follows that if $x \in [a]$, then xRa, and the fact that aRb combined with transitivity of R impliess xRb, so that $x \in [b]$. Hence $[a] \subseteq [b]$. Likewise, if $x \in [b]$, then xRb, and the fact that aRb combined with symmetry and transitivity of R implies xRa, so that $x \in [a]$. Hence $[b] \subseteq [a]$. Therefore $[a] = [b]$.

Finally, we observe that since each equivalence class of R is a subset of A, we have $\bigcup_{X \in A/R} X \subseteq A$. Furthermore, for any $a \in A$, reflexivity of R implies $a \in [a] \subseteq \bigcup_{X \in A/R} X$. Thus $A \subseteq \bigcup_{X \in A/R} X$. We conclude $\bigcup_{X \in A/R} X = A$.

For the converse, let $\mathscr{P}$ be a partition of A. Define a relation R on A as follows: If $a, b \in A$, then aRb if and only if there exists $X \in \mathscr{P}$ with $a \in X$ and $b \in X$. Let us first show R is an equivalence relation.

<u>Reflexive</u>: Let $a \in R$. Since $\mathscr{P}$ is a partition, $\bigcup_{X \in \mathscr{P}} X = A$. Thus there exists $X \in \mathscr{P}$ such that $a \in X$. Hence aRa.

<u>Symmetric</u>: Let $a, b \in A$ with aRb. Then there exists $X \in \mathscr{P}$ such that $a \in X$ and $b \in X$. But then $b \in X$ and $a \in X$, so that bRa.

<u>Transitive</u>: Let $a, b, c \in A$ with aRb and bRc. Since aRb, there exists $X \in \mathscr{P}$ such that $a \in X$ and $b \in X$. Likewise, since bRc, there exists $Y \in \mathscr{P}$ such that $b \in Y$ and $c \in Y$. Since $b \in X \cap Y$, we conclude $X \cap Y \neq \emptyset$, and the fact the sets in $\mathscr{P}$ are pairwise disjoint implies that $X = Y$. This $a \in X$ and $c \in Y = X$, so that aRc.

Next, we shall show that $A/R = \mathscr{P}$. Let $[a] \in A/R$. Since aRa, there exists $X \in \mathscr{P}$ such that $a \in X$. Furthermore, since the sets in $\mathscr{P}$ are pairwise disjoint, for any $b \in A$ we have $b \in [a]$ if and only if aRb if and only $b \in X$. Thus $[a] = X$. Hence $A/R \subseteq \mathscr{P}$. For the reverse inclusion, let $X \in \mathscr{P}$. Since the sets in $\mathscr{P}$ are nonempty, there exists $a \in X$. Because the sets in $\mathscr{P}$ are pairwise disjoint, for any $b \in A$ we have $b \in X$ if and only if aRb if and only $b \in [a]$. Thus $X = [a]$. Hence $\mathscr{P} \subseteq A/R$. We conclude $A/R = \mathscr{P}$. □

1.3 Order Relations

In the previous section, we discussed equivalence relations, which generalize the equality relation $=$. In this section, we shall introduce order relations, which generalize the subset relation $\subseteq$ and the less than or equal relation $\leq$.

Definition 1.21 Let A be a set, and let R be a relation on R. We define the following property for the relation R:

Antiymmetric: For all $x, y \in A$, if xRy and yRx, then $x = y$.

We call R a *partial order* if R is reflexive, antisymmetric, and transitive. In this case, the pair (A, R) consisting of the set A and the partial order R is called a *partially ordered set* (or a *poset* for short).

■ **Example 1.22** Let X be a set and consider the power set $\mathscr{P}(X)$. The subset relation $\subseteq$ is a partial order on $\mathscr{P}(X)$. ■

Note that in Example 1.22 two elements need not be related. For example, if $X = \{1,2,3\}$, then we see that the sets $A = \{1,2\}$ and $B = \{2,3\}$ in $\mathscr{P}(X)$ are not related, since neither is a subset of the other. This motivates the following definitions.

Definition 1.23 Let A be a set, and let R be a partial order on A. We say that two elements $a, b \in A$ are *comparable* if either aRb or bRb. When a and b are not comparable, we say they are *incomparable*.

Definition 1.24 Let A be a set, and let R be a relation on R. We define the following property for the relation R:

Total: For all $x, y \in A$, either xRy or yRx.

In other words, a relation is total if any two elements are comparable. We call R a *total order* if R is reflexive, antisymmetric, transitive, and total. Thus a total order is precisely a partial order that is total, and every total order is a partial order. Alternately, one can describe a total order as a partial order for which every pair of elements is comparable. When the relation R is a total order, we say that the pair (A, R) is a *totally ordered set*.

■ **Example 1.25** Let $\mathbb{R}$ be the set of real numbers with the usual less than or equal to relation $\leq$. Then $\leq$ is a total order, and $(\mathbb{R}, \leq)$ is a totally ordered set. ■

★ We have seen that equivalence relations generalize equality. In a similar manner, the subset relation of Example 1.22 is the prototypical example of a partial order, and the less than or equal relation of Example 1.25 is the prototypical example of a total order.

Type of Relation	Meant to Generalize
Equivalence Relation	Equality $=$
Partial Order	Subset $\subseteq$
Total Order	Less Than or Equal $\leq$

When R is a partial order or a total order, it is common to use symbols such as $\subseteq$, $\leq$, $\preceq$, or $\lesssim$ in place of R. It is also common to use the shortened term "order" to mean either a partial order or a total order, so when one see an author refer to an "order" it important as a reader to be aware of context and ascertain whether it is a partial order or a total order being referred to.

■ **Example 1.26** Consider the set $\mathbb{N}$, and define a relation $\leq_d$ on $\mathbb{N}$ by

$$a \leq_d b \qquad \text{if and only if } a \mid b.$$

Let us show that $\leq_d$ is a partial order on $\mathbb{N}$.
Reflexive: Let $a \in \mathbb{N}$. Then $a = 1 \cdot a$, so $a \mid a$.
Antisymmetric: Let $a, b \in \mathbb{N}$, and suppose that $a \mid b$ and $b \mid a$. Then there exist $m, n \in N$ such that $b = ma$ and $a = nb$. Hence $b = ma = m(na_{=}(mn)a$, and since $a \neq 0$, we may cancel to obtain $1 = mn$. Since m and n are natural numbers, we conclude that $m = n = 1$. Thus $a = b$.
Transitive Let $a, b, c \in \mathbb{N}$, and suppose that $a \mid b$ and $b \mid c$. Then there exist $m, n \in N$ such that $b = ma$ and $c = nb$. Hence $c = nb = n(ma) = (nm)a$, and $a \mid c$. ■

★ Suppose that we extend $\leq_d$ to Z using the same definition: $a \leq_d b$ if and only if $a \mid b$. Observe that $\leq_d$ is not a partial order on $\mathbb{Z}$. Although, $\leq_d$ is reflexive and transitive on $\mathbb{Z}$, it is not antisymmetric; for example $3 \mid -3$ and $-3 \mid 3$, but $3 \neq -3$.

■ **Example 1.27** If A is any set, the equality relation $=$ is a partial order. Note that if $a, b \in A$ with $a \neq b$, then a and b are incomparable. Therefore the equals relation $=$ is the partial order on A with the fewest possible pairs of comparable elements. ■

1.3.1 Successors, Predecessors, Maximal, and Minimal Elements

Definition 1.28 Let R be a partial order on a set A. If $a, b \in A$ have the property that

"For any $c \in A$, if aRc and cRb, then $c = a$ or $c = b$."

then we say that b *covers* a. Furthermore, when b covers a, we also say that a is an *immediate predecessor* of b, and that b is an *immediate successor* of a.

Definition 1.29 Let R be a partial order on a set A. If $a \in A$ has the property that for any $b \in A$, bRa implies $b = a$, we say that a is *minimal*. A partially ordered set may have no, one, or multiple minimal elements. When a partially ordered set has a unique minimal element, we call that element a *minimum*.

Likewise, if $a \in A$ has the property that for any $b \in A$, aRb implies $b = a$, we say that a is *maximal*. A partially ordered set may have no, one, or multiple maximal elements. When a partially ordered set has a unique maximal element, we call that element a *maximum*.

1.3.2 Hasse Diagrams

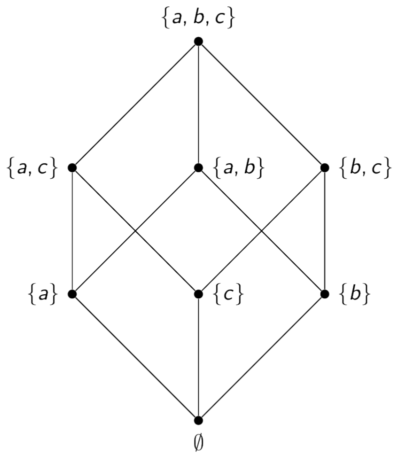
A *Hasse diagram* provides a method for drawing a picture to visualize a partially ordered set with a finite number of elements. Suppose that $(A, \leq)$ is a partially ordered set and that A has a finite number of elements. To create a Hasse diagram, each element of the set is written in the plane (either as the element itself or as a "dot" labeled by the element) and one draws a line segment (straight or curved) from an element a to an element b if and only if b covers a. These segments may cross each other but must not touch any elements (or "dots") other than at their endpoints.

■ **Example 1.30** Let $X = \{a, b, c\}$. Then

$$\mathscr{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$$

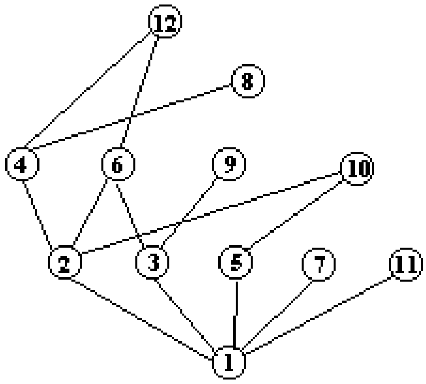
and we may make $\mathscr{P}(X)$ into a partially ordered set with the subset relation $\subseteq$.

To draw the Hasse diagram for $(\mathscr{P}(X), \subseteq)$, we note that the empty set is the unique minimal element. Its immediate successors are the singleton subsets containing exactly one element each, the immediate successors of the singleton subsets are the subsets containing exactly two element, and the (only) immediate successor of those is the set $\{a, b, c\}$ with three elements.



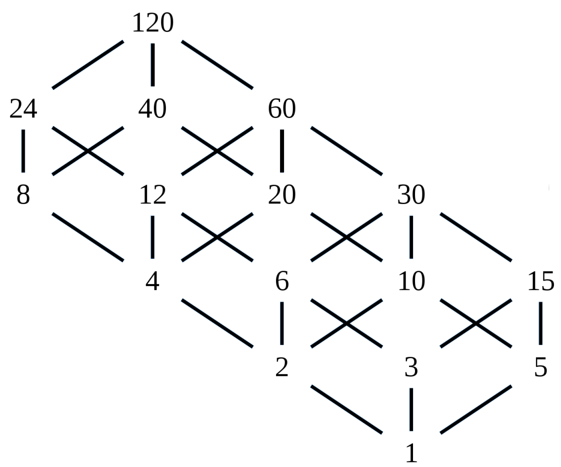
We see that $\emptyset$ is the unique minimal element — or minimum — of $\mathscr{P}(X)$, and $\{a,b,c\}$ is the unique maximal element — or maximum — of $\mathscr{P}(X)$. ■

■ **Example 1.31** Let $A = \{1,2,\ldots,11,12\}$ be the set of natural numbers from 1 to 12 with the divisibility partial order $\leq_d$. The Hasse diagram for this partially ordered set is the following:



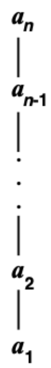
We see that 1 is the unique minimal element — or minimum — of this partially ordered set. and the elements 8, 10, 11, and 12 are maximal elements of this partially ordered set. ■

■ **Example 1.32** Let $A = \{1,2,3,5,4,6,8,10,12,15,20,24,30,40,60,120\}$ be the set of natural number divisors of 120 with the divisibility partial order $\leq_d$. The Hasse diagram for this partially ordered set is the following:



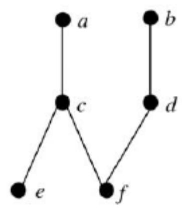
■

■ **Example 1.33** Let $(A, \leq$ be a totally ordered set with A finite. Since A is finite and totally ordered, we may write the elements of A as $A = \{a_1, \ldots a_n\}$ with $a_i \leq a_{i+1}$ for all $1 \leq i \leq n-1$. The Hasse diagram is the following:



Thus any finite totally ordered set will have a Hasse diagram that is simply a single vertical path. ■

■ **Example 1.34** Hasse diagrams not only provide a method for picturing a given partially ordered set, they can also be used to define a partially ordered set. For instance, suppose we draw the following Hasse diagram



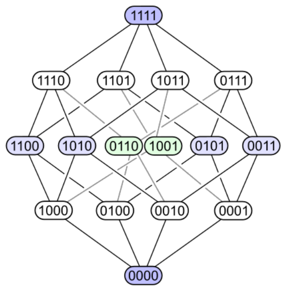
This Hasse diagram defines a partially ordered set as follows: The underlying set has one element for each vertex; i.e., $A = \{a, b, c, d, e, f\}$. Furthermore, by transitivity, an element x is related to (i.e., less than or equal to) another element y if and only if $x = y$ or there is a path of segments starting at x and going up to y. In the Hasse diagram above, we have only the following pairs related:

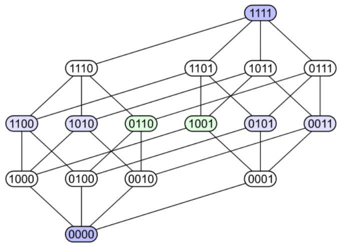
$$e \leq e, \quad e \leq c, \quad e \leq a, \quad f \leq f, \quad f \leq c, \quad f \leq a, \quad f \leq d, \quad f \leq b,$$

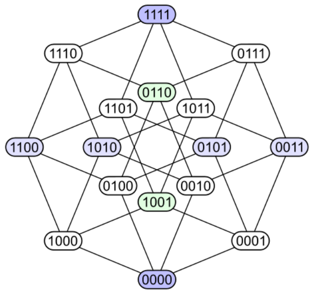
$$c \leq c, \quad c \leq a, \quad a \leq a, \quad d \leq d, \quad d \leq b, \quad b \leq b$$

Notice that drawing the Hasse diagram to define the partially ordered set $(A, \leq)$ is significantly easier than describing the set and the related pairs. ■

The method of drawing Hasse diagrams involves choices of where to place the elements, and the different choices can result in pictures of the same diagram that, while equivalent, look very different to the eye. One's first attempt at drawing a diagram often produces a poor result, and it is typical that one uses a first attempt to reposition the vertices and segments to draw a “nicer" version of the Hasse diagram. The term “nicer" can take on different meanings in different contexts. For instance, consider the power set of a set with four elements partially ordered by the subset relation. The following three pictures are all Hasse diagrams for this partially ordered set:







The first Hasse diagram is useful for emphasizing the levels of the partially ordered set, showing how many steps one has to take from the minimal element to reach a given element. In the second diagram, by making some segments longer than others, one can see a 4-dimensional cube written as a combinatorial union of two 3-dimensional cubes. The third diagram emphasizes the internal symmetry of the partially ordered set. While each of these diagrams represent the same partially ordered set, in a specific situation one diagram may be more useful than another for emphasizing some particular aspect of the partially ordered set.

1.3.3 Least Upper Bounds and Greatest Lower Bounds

Definition 1.35 Let (A,R) be a partially ordered set, and let B be a subset of A. We say that an element $a \in A$ is an *upper bound* for B (or a *supremum* of B) if for every $b \in B$ we have bRa. We say that an element $a \in A$ is a *least upper bound* if

(i) a is an upper bound for B, and

(i) whenever a' is an upper bound for B, we have aRa'.

When it exists, we write $\sup B$ for the least upper bound of B.

Definition 1.36 Let (A,R) be a partially ordered set, and let B be a subset of A. We say that an element $a \in A$ is an *lower bound* for B (or *infimum* for B) if for every $b \in B$ we have aRb. We say that an element $a \in A$ is a *greatest bound* if

(i) a is a lower bound for B, and

(i) whenever a' is a lower bound for B, we have $a'Ra$.

When it exists, we write $\inf B$ for the greatest lower bound of B.

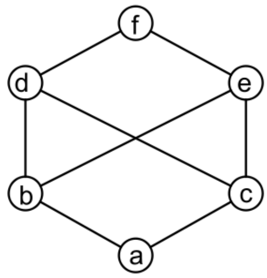
■ **Example 1.37** Let X be a set, and let $\mathscr{P}(X)$ denote its power set wit the partial order give by the subset relation $\subseteq$. If $\mathscr{B} \subseteq \mathscr{P}(X)$, define $S := \bigcup_{A \in \mathscr{B}} A$. We shall show that S is the least upper bound (or supremum) of $\mathscr{B}$.

First, if $B \in \mathscr{B}$, then $B \subseteq \bigcup_{A \in \mathscr{B}} A = S$, so S is an upper bound of $\mathscr{B}$. Second, if S' is an upper bound of $\mathscr{B}$, then $A \subseteq S'$ for all $A \in \mathscr{B}$. Hence $\bigcup_{A \in \mathscr{B}} A \subseteq S'$, and $S \subseteq S'$. Thus S is the least upper bound (or supremum) of $\mathscr{B}$. ■

One can easily see that if a least upper bound exists, then it is is unique: If a and a' are least upepr bounds for a set B, then since a and a' are also both upper bounds for B, we conclude that aRa' and $a'Ra$. Antisymmetry then implies $a = a'$.

However, it is possible that a least upper bound may not exist. Of course, if a subset has no upper bounds, then there can be no *least* upper bound. But, more importantly, least upper bounds can fail to exist even for subsets that do have an upper bound.

Suppose (A,R) is a partially ordered set, and $B \subseteq A$. Suppose that B has at least one upper bound in A. There are two ways that B can fail to have a least upper bound: One possible reason for the failure is that there may be lower bounds that are not comparable; for instance in the partially ordered set with Hasse diagram



the set $B = \{b, c\}$ has upper bounds d, e, and f. However, the elements d and e are not comparable, so B has no least upper bound. The second possible reason for the failure is that for each upper bound there may another upper bound that is strictly smaller. For example, if we take the rational number $\mathbb{Q}$ with the usual ordering $\leq$, and let $B := \{x \in \mathbb{Q} : x^2 < 2\}$, then (keeping in mind that $\sqrt{2}$ is irrational) we see that if $a \in \mathbb{Q}$ is any upper bound of B, then $\sqrt{2} < a$, and by choosing $a' \in \mathbb{Q}$ with $\sqrt{2} < a' < a$, we see that a' is an upper bound of B smaller than a. Thus B has no *least* upper bound. Note that if (A, R) is a totally ordered set, then a subset $B \subseteq A$ that has an upper bound can only fail to have least upper bound for the second reason given above.

Similar statements and examples can be given for greatest lower bounds. In particular, a subset of a partially ordered set may fail to have a great lower bound (even if it has lower bounds), but if a greatest lower bound exists, it is unique.

■ **Example 1.38** Let $\mathbb{R}$ denote the set of real numbers with the usual order $\leq$. Consider the following sets

$$S := [0, 5), \qquad T := \{1/n : n \in \mathbb{N}\}, \qquad \text{and} \qquad U := \{1, 2, 3, 4, \ldots\}.$$

We see that $\sup S = 5$ and $\inf S = 0$. Also, $\sup T = 1$ and $\inf T = 0$. Finally, we have $\inf U = 1$ and $\sup U$ does not exist. ■

★ Consider the set of real numbers $\mathbb{R}$ with the usual order $\leq$. It is a fact that if B is a nonempty subset of $\mathbb{R}$ that has an upper bound, then there exists a least upper bound for B in $\mathbb{R}$. This is called the *least upper bound property* of the real numbers. The real numbers are a field (i.e., one can add, subtract, multiply, and divide by nonzero numbers), and the usual order $\leq$ on $\mathbb{R}$ is a total order that respects the addition and multiplication by positive elements. Such a pair $(\mathbb{R}, \leq)$ is called an ordered field. The real numbers are characterized as the unique ordered field with the least upper bound property.