Important Theorems from Calculus

In statements of the following results, recall that if $a, b \in \mathbb{R}$ with $a < b$, then $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ and $(a, b) := \{x \in \mathbb{R} : a < x < b\}$.

(EVT) Extreme Value Theorem
If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded on $[a, b]$ and f achieves both a maximum value on $[a, b]$ and a minimum value on $[a, b]$.

(IVT) Intermediate Value Theorem
If $f : [a, b] \to \mathbb{R}$ is continuous, and z is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = z$.

(MVT) Mean Value Theorem
If $f : [a, b] \to \mathbb{R}$ is continuous, f is differentiable at every point in (a, b), then there exists a number $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(FTC-V1) Fundamental Theorem of Calculus — Version I
If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable and

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

(FTC-V2) Fundamental Theorem of Calculus — Version II
If $f : [a, b] \to \mathbb{R}$ is differentiable and its derivative $f' : [a, b] \to \mathbb{R}$ is continuous, then f' is Riemann integrable and

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

In addition, the following useful facts are consequences of these theorems:

Fact 1: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all x in $[a, b]$, then f is constant on $[a, b]$.

Fact 2: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) > 0$ for all x in $[a, b]$, then f is strictly increasing on $[a, b]$.

Fact 3: If $f : \mathbb{R} \to \mathbb{R}$ is continuous and f is injective on $[a, b]$, then f is either strictly increasing or strictly decreasing on $[a, b]$.

Definition of a Field

If S is a set, a *binary operation* is a function from $S \times S$ to S. In other words, a binary operation takes a pair of elements from S as input and assigns another element of S as an output. We usually use arithmetic operations, such as $+$ or $\cdot$, for a binary operation. We write $a + b$ for the value that $+$ assigns to the pair (a, b), and we write $a \cdot b$ for the value $\cdot$ assigns to the pair (a, b).

A *field* is a set $\mathbb{F}$ together with two binary operations $+$ and $\cdot$ satisfying the following axioms:

(1) (Associativity of $+$) For all $a, b, c \in \mathbb{F}$, we have $(a+b)+c = a+(b+c)$.
(2) (Existence of Additive Identity) There exists an element $0 \in \mathbb{F}$ such that $0 + a = a$ for all $a \in \mathbb{F}$.
(3) (Existence of Additive Inverses) For all $a \in \mathbb{F}$ there exists $-a \in \mathbb{F}$ such that $a + (-a) = 0$.
(4) (Commutativity of $+$) For all $a, b \in \mathbb{F}$, we have $a + b = b + a$.
(5) (Associativity of $\cdot$) For all $a, b, c \in \mathbb{F}$, we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(6) (Existence of Multiplicative Identity) There exists an element $1 \in \mathbb{F} \setminus \{0\}$ such that $1 \cdot a = a$ for all $a \in \mathbb{F}$.
(7) (Existence of Multiplicative Inverses) For all $a \in \mathbb{F} \setminus \{0\}$ there exists $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$.
(8) (Commutativity of $\cdot$) For all $a, b \in \mathbb{F}$, we have $a \cdot b = b \cdot a$.
(9) (Distributivity of Multiplication over Addition) For all $a, b, c \in \mathbb{F}$, we have $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.