Assignment #2

The following homework problems are **due at the beginning of class on Wednesday, September 15**. Please follow all guidelines as described in the "Homework" section of the course syllabus. Also remember that homework can and should be worked on and discussed with others, however, the write-up should be independent.

1. Suppose that $\{x_i\}_{i\in I}$ is a collection of non-negative real numbers indexed by the set I. (We make no assumptions on the cardinality of I, and in particular I is allowed to be uncountable.) Define

$$\sum_{i\in I} x_i := \sup\left\{\sum_{i\in F} x_i \ : \ F \text{ is a finite subset of } I\right\}.$$

Prove that if $\sum_{i\in I} x_i < \infty$, then $\{i \in I : x_i \neq 0\}$ is countable. (Recall that *countable* means finite or countably infinite.)

2. Let $p \in [1, \infty)$, and consider

$$\ell^p := \{(x_1, x_2, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

with the p-norm

$$\|(x_i)_{i=1}^{\infty}\|_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}.$$

Assume that $(\ell^p, \|\cdot\|_p)$ is a normed space; in particular, you may assume ℓ^p is a vector space and $\|\cdot\|_p$ satisfies all the properties of a norm. Prove that $(\ell^p, \|\cdot\|_p)$ is complete.

3. Let H be a Hilbert space. Prove that the unit ball

$$\mathrm{Ball}(H) := \{\vec{x} \in H : \|\vec{x}\| \leq 1\}$$

is compact if and only if H is finite dimensional.

4. Let H be a Hilbert space. Prove that H is separable if and only if H has a countable basis. (Recall that *countable* means finite or countably infinite. Also recall that a topological space is *separable* if it has a countable dense subset.)

5. Let H be an infinite-dimensional Hilbert space. Prove that any Hamel basis of H cannot be orthonormal. (Hint: Suppose β is a Hamel basis that is orthonormal. Choose a sequence $(\vec{e}_n)_{n=1}^{\infty} \subseteq \beta$ and show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n} \vec{e}_n$ converges to an element in H. Use this to obtain a contradiction.)

A Note Regarding the Write-Up of Proofs: When writing proofs, you will often need to first do work on scratch paper and then write up your final solution once you have figured out how to do the proof. You should only hand in the final result — not the scratchwork. The write-up that you hand in should be written neatly and legibly. Also, when you are writing proofs remember that the standard rules of English usage still apply; in particular, you should use complete sentences and proper punctuation. As in any class that requires writing assignments, your grade will be based in part on your ability to write clearly, convincingly, and correctly.

When writing up proofs, you should write out the statement you are proving followed by your proof. For example, if a problem asked you to show that A and B imply C, then you should write

Claim: If A and B, then C.

Proof: Since A and B hold we see that . . .

$\square$

By doing this, you begin with a statement of your assumptions and with a carefully worded statement of the result to be proven. This will help you keep track of what you are allowed to assume and what you are trying to show.