Exam 1

The following are due at the beginning of class on Friday, Sept. 26.

Problem 1: Let X be a vector space, and suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on X and that $\mathcal{T}_1$ and $\mathcal{T}_2$ are the corresponding topologies determined by these norms. Prove that if X is complete in both norms and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$.

Problem 2: Let X be a Hilbert space, and let $\mathcal{E}$ be an orthonormal basis for X. Prove that if $\{h_n\}_{n=1}^{\infty}$ is a sequence of vectors in X, then $\lim_{n\to\infty}\langle h_n, x\rangle = 0$ for all $x \in X$ if and only if $\sup\{\|h_n\| : n \in \mathbb{N}\} < \infty$ and $\lim_{n\to\infty}\langle h_n, e\rangle = 0$ for all $e \in \mathcal{E}$.

Problem 3: Let X be a normed space, let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in X, and let $z_1, \ldots, z_n \in \mathbb{C}$. Prove that there exists a bounded linear functional $f : X \to \mathbb{C}$ such that $f(x_i) = z_i$ for all $1 \leq i \leq n$.

Problem 4: Let X and Y be Banach spaces, and suppose that $T : X \to Y$ is a bounded linear map. Prove that there exists $c > 0$ such that $\|T(x)\| \geq c\|x\|$ for all $x \in X$ if and only if $\ker T = \{0\}$ and $\operatorname{Ran} T$ is closed.

Problem 5: Let M be the following subspace of the Banach space ℓ^∞:

$$M := \{(x_1, x_2, \ldots) \in \ell^\infty : x_i = 0 \text{ for all but finitely many } i\}.$$

What is the closure of M? Prove that the exists a bounded linear functional $f : \ell^\infty \to \mathbb{C}$ such that $f(x) = 0$ for all $x \in M$ and $f(1, 1, 1, \ldots) = 1$.