Assignment 1

The following problems are due at the beginning of class on Friday, Sept. 19.

Problem 1: Let $(X, \|\cdot\|)$ be a normed space. Prove that $\|\cdot\|$ is the norm induced from an inner product if and only if $\|\cdot\|$ satisfies the *parallelogram law*:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad \text{for all } x, y \in X.$$

Problem 2: Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Prove that the closed unit ball of X is compact if and only if X is finite dimensional.

Problem 3: Let X be a vector space. Prove that all norms on X are equivalent if and only if X is finite dimensional.

Problem 4: Recall that a topological space is defined to *separable* if it has a countable dense subset. Prove that a Hilbert space $(X, \langle \cdot, \cdot \rangle)$ is separable if and only if it has a countable orthonormal basis.

Problem 5: Prove that if X is an infinite-dimensional Banach space, then any vector space basis of X is uncountable. (Hint: Use the Baire category theorem.) (This result shows that any Banach space has vector space dimension that is either finite or uncountable.)