Assignment 2

The following are due at the beginning of class on Friday, Oct. 24.

Problem 1: Ideal structure of $B(H)$.

Definition: A subset $I \subseteq B(H)$ is called an *ideal* if I is a subspace and whenever $T \in I$ and $S \in B(H)$, then $ST \in I$ and $TS \in I$. An ideal is called a *closed ideal* if it is closed in the operator norm topology on $B(H)$.

(a) (2 points) Let H be a Hilbert space. For $h, k \in H$, define $\Theta_{h,k} : H \to H$ by $\Theta_{h,k}(x) := \langle x, k \rangle h$. Prove that $T : H \to H$ is a bounded rank-one operator on H if and only if $T = \Theta_{h,k}$ for some nonzero vectors $h, k \in H$.

(b) (2 points) Let H be a Hilbert space, let $\mathcal{F}(H)$ denotes the bounded finite-rank operators on H, and let $\mathcal{K}(H)$ denote the compact operators on H. Prove that

$$\mathcal{F}(H) = \operatorname{span}\{\Theta_{h,k} : h, k \in H\} \quad \text{and} \quad \mathcal{K}(H) = \overline{\operatorname{span}}\{\Theta_{h,k} : h, k \in H\}.$$

(c) (2 points) Let H be a Hilbert space. Prove that if I is any (not necessarily closed) nonzero ideal of $B(H)$, then $\mathcal{F}(H) \subseteq I$.

(d) (2 points) Prove that a projection $P \in B(H)$ is compact if and only if $\operatorname{rank} P < \infty$.

(e) Later in the course, we shall prove that if I is a (not necessarily closed) ideal of $B(H)$ and I contains a non-compact operator, then there exists a projection $P \in I$ with $\operatorname{rank} P = \infty$. For now, simply marvel at the beauty of this fact.

(f) (2 points) Using parts (a)–(e) above, prove the following:

Let H be a separable infinite-dimensional Hilbert space. If I is a (not necessarily closed) ideal of $B(H)$ that is nonzero and proper, then $\mathcal{F}(H) \subseteq I \subseteq \mathcal{K}(H)$. In addition, $\mathcal{K}(H)$ is the unique proper nonzero closed ideal of $B(H)$.

Remark: When H is separable and infinite dimensional, $\mathcal{K}(H)$ is the unique closed nonzero, proper ideal of $B(H)$. However, there are in general many non-closed ideals strictly between $\mathcal{F}(H)$ and $\mathcal{K}(H)$. Look up the trace-class operators and the Hilbert-Schmidt operators for some examples.